ESTIMATE OF LIMIT LOAD OF ELASTIC-PLASTIC SHELLS OF REVOLUTION

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A number of papers [1-4] have appeared on the numerical solution of problems of determining the axisymmetric stress-strain state of elastic-plastic shells of revolution. The solution requires the evaluation of integrals of the stress over the thickness of the shell, which increases the amount of processing and storage of information on the stress-strain state as compared with the elastic solution.

Approximate relations were derived in [5, 6] for the elastic-plastic flow of shells under the Mises yield condition based on the approximation of a finite ratio between forces and moments. By using these relations the volume of information required for storage in the computer memory is appreciably reduced, since it is no longer necessary to evaluate integrals over the thickness of the shell. In [7, 8] the approximate yield surface derived in [5, 6] was used to determine the limit loads of shells under tensile stress. Good agreement was noted between the limit loads found by using the approximate and Mises yield surfaces.

Our numerical experiments showed that the limit loads of shells of revolution under axisymmetric loading by an external hydrostatic pressure can be determined accurately enough for practical purposes by using the approximate equations derived in [5, 6] which relate directly the rate of change of forces and moments with the strain rates and the changes of the curvature of the middle surface of the shell.

1. Controlling Relations of the Elastic-Plastic Axisymmetric Deformation of Shells. For an ideally elastic-plastic material in an axisymmetric plane stressed state the controlling relations connecting the stress rates $\dot{\sigma}_i$ and the strain rates $\dot{e}_i$ (a dot over a quantity denotes differentiation with respect to a certain strain parameter) have the form

$$\begin{gathered}\dot{\sigma}_1 = \gamma A_{ij}\dot{e}_j, \quad A_{ii} = 1 - cf_i^2/P, \quad A_{ij} = \nu - cf_if_j/P \quad (i \neq j),\\ c = \begin{cases}1, & \text{if} \quad J = 1/3 \text{ and } \dot{\omega} > 0,\\ 0, & \text{if} \quad J < 1/3 \quad \text{or} \quad J = 1/3 \text{ and } \dot{\omega} \leqslant 0,\end{cases}\\ J = (\sigma_1')^2 + (\sigma_2')^2 + \sigma_1'\sigma_2', \quad \dot{\omega} = f_i\dot{e}_i,\\ f_1 = \sigma_1' + \nu\sigma_2', \quad \sigma_1' = (2\sigma_1 - \sigma_2)/3 \quad (1 \rightleftarrows 2),\\ P = (\sigma_1')^2 + (\sigma_2')^2 + 2\nu\sigma_1'\sigma_2', \quad \gamma = E/[\sigma_T(1 - \nu^2)],\end{gathered} \tag{1.1}$$

where E is Young's modulus, ν is Poisson's ratio, the σ_i are the stress components relative to the yield point under uniaxial tension (compression) σ_T; from now on, unless specially stipulated, the subscripts take on the values 1 and 2, and summation is performed over repeated indices.

By using the Kirchhoff–Love kinematic hypotheses, the strain rates in the shell are expressed by the formulas

$$\dot{e}_i = \dot{\varepsilon}_i + 2\zeta\dot{k}_i, \tag{1.2}$$

where ε_i and k_i are, respectively, the strains and the changes of curvature of the middle surface (k_i and $K_ih/4$, and the K_i are the corresponding dimensional quantities); $\xi = 2z/h$; z is the distance along the normal from the middle surface of the shell, $-h/2 \leqslant z \leqslant h/2$; h is the thickness of the shell.

The forces n_i and moments m_i ($n_i = N_i/\sigma_T h$, $m_i = 4M_i/(\sigma_T h^2)$, where N_i and M_i are the dimensional forces and moments) in the middle surface of the shell are calculated from the

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formulas

$$n_i = \frac{1}{2}\int_{-1}^{1} \sigma_i d\zeta, \quad m_i = \int_{-1}^{1} \sigma_i \zeta d\zeta. \tag{1.3}$$

The relation between the rates of change of forces and moments and the strain rates in the middle surface of the shell can be written in the form

$$[\dot{n}_1, \dot{n}_2, \dot{m}_1, \dot{m}_2]^T = [B_{ij}][\dot{\varepsilon}_1, \dot{\varepsilon}_2, \dot{k}_1, \dot{k}_2]^T \quad (i, j = 1, 2, 3, 4), \tag{1.4}$$

where the superscript T denotes transposition.

It follows from Eqs. (1.1)-(1.3) that the elements of the symmetric matrix $[B_{ij}]$ are given by the expressions

$$B_{i+2k, j+2l} = 2^{(k+l-1)}\gamma \int_{-1}^{1} A_{ij}\zeta^{(k+l)} d\zeta \quad (i, j = 1, 2;\ k, l = 0, 1). \tag{1.5}$$

The equations derived by Ivanov in [6] directly connect the rates of change of forces and moments with the strain rates of the middle surface of the shell. In the derivation of these equations it was assumed that an element of the shell is either completely in a plastic state or completely in an elastic state, from which it follows that in this case there is a finite ratio between the forces and moments. In [5] Ivanov proposed an approximation of this ratio which he used in [6] to construct equations connecting directly the rates of change of forces and moments with the strain rates of the middle surface of the shell. For the asisymmetric deformation of a shell these equations can be written in the form (1.4), where

$$\begin{gathered}
B_{11} = \gamma\left(1 - \alpha\frac{\gamma_1^2}{S}\right), \quad B_{12} = \gamma\left(\nu - \alpha\frac{\gamma_1\gamma_2}{S}\right), \\
B_{13} = -\frac{4}{3}\gamma\alpha\frac{\theta_1\gamma_1}{S}, \quad B_{14} = -\frac{4}{3}\gamma\alpha\frac{\theta_2\gamma_1}{S}, \quad B_{22} = \gamma\left(1 - \alpha\frac{\gamma_2^2}{S}\right), \\
B_{23} = -\frac{4}{3}\gamma\alpha\frac{\theta_1\gamma_2}{S}, \\
B_{24} = -\frac{4}{3}\gamma\alpha\frac{\theta_2\gamma_2}{S}, \quad B_{33} = \frac{4}{3}\gamma\left(1 - \alpha\frac{4}{3}\frac{\theta_1^2}{S}\right), \\
B_{34} = \frac{4}{3}\gamma\left(\nu - \alpha\frac{4}{3}\frac{\theta_2\theta_1}{S}\right), \quad B_{44} = \frac{4}{3}\gamma\left(1 - \alpha\frac{4}{3}\frac{\theta_2^2}{S}\right), \\
\alpha = \begin{cases} 1, & \text{if} \quad f = 1 \text{ and } \dot{\Omega} > 0, \\ 0, & \text{if} \quad f < 1 \quad \text{or} \quad f = 1 \text{ and } \dot{\Omega} \leqslant 0. \end{cases} \\
\dot{\Omega} = \gamma_i\dot{\varepsilon}_i + \frac{4}{3}\theta_i\dot{k}_i, \quad \gamma_1 = \alpha_1 + \nu\alpha_2, \quad \theta_1 = \beta_1 + \nu\beta_2 \quad (1 \rightleftarrows 2), \\
\alpha_i = \frac{\partial f}{\partial n_i}, \quad \beta_i = \frac{\partial f}{\partial m_i}, \quad S = \gamma_i\alpha_i + \frac{4}{3}\theta_i\beta_i, \\
f = Q_n + \frac{1}{2}Q_m - \frac{0.25\left(Q_nQ_m - Q_{nm}^2\right)}{Q_n + 0.48Q_m} + \frac{1}{2}\sqrt{Q_m^2 + 4Q_{nm}^2}, \\
Q_n = n_1^2 - n_1n_2 + n_2^2, \quad Q_m = m_1^2 - m_1m_2 + m_2^2, \\
Q_{nm} = n_1m_1 - \frac{1}{2}(n_1m_2 + n_2m_1) + n_2m_2.
\end{gathered} \tag{1.6}$$

2. Formulation of the Problem and Method of Solution. The formulation and method of numerical solution of the problem of the elastic-plastic axisymmetric deformation of shells of revolution are described in [4].

We take the Sanders geometrical nonlinear equations of shells with small deformations and moderate deflections as a basis. For shells of revolution under axisymmetric deformation the fundamental system of equations is reduced to a set of six ordinary nonlinear first order differential equations for the required rates of change of displacements and forces in the middle surface of the shell. Three boundary conditions are set on each end of the shell.

In the numerical solution the derivatives with respect to the meridional coordinate are approximated by second order central differences. For constant coefficients the difference equations form a system of linear algebraic equations which is solved by the matrix pivotal method.

This is a Cauchy problem with respect to the deformation parameter. The second order predictor-corrector method [2] is used in integrating with respect to the deformation parameter. Because of the geometrical and physical nonlinearities, iteration was performed with respect to the deformation parameter at each step.

The axial displacement of the end of the shell was taken as the deformation parameter. The limit load is defined as the maximum on the graph of the external force as a function of the axial displacement.

3. Numerical Results. Using the algorithm described in Sec. 2, limit loads were calculated for a shell consisting of cylindrical and conical parts acted upon by an external hydrostatic pressure (Fig. 1). The calculations were performed for a shell with the following parameters: $R/h = 100$, $L_{cy}/L_{co} = 1$, $L/R = 0.2$, $E/\sigma_T = 250$, $\nu = 0.3$; the angle θ was varied from 90 to 130°. Here R is the radius of the middle surface of the cylindrical part of the shell, L_{cy} and L_{co} are respectively the lengths of meridians of the cylindrical and conical portions; $L = L_{cy} + L_{co}$ is the total length of a meridian of the middle surface of the shell; θ is the angle between the axis of revolution and the normal to the middle surface of the conical part of the shell.

We present the results of calculations for the following boundary conditions:

$$\dot{n}_1 = b^2 R\dot{p}/2h, \quad \dot{w} = \dot{\Phi} = 0 \quad \text{for} \quad s = 0, \tag{3.1}$$

$$\dot{u} = \dot{w} = \dot{\Phi} = 0 \quad \text{for} \quad s = L, \tag{3.2}$$

where $w = W/h$, $u = U/L$, W is the displacement along the normal to the middle surface of the shell, U is the displacement along a meridian, Φ is the angle of rotation of the normal to the middle surface of the shell, s is the arc length along a meridian ($0 \leqslant s \leqslant L$), $p = P/(b^2 \sigma_T)$, P is the hydrostatic pressure, and $b = h/L$.

Boundary conditions (3.1) and (3.2) correspond to the fact that the left-hand end of the shell is displaced in the axial direction under an axial force n_1, and the right-hand end is clamped. With respect to the angle of rotation of the normal and the displacement in the direction of the normal to the middle surface, the boundary conditions correspond to both ends being clamped.

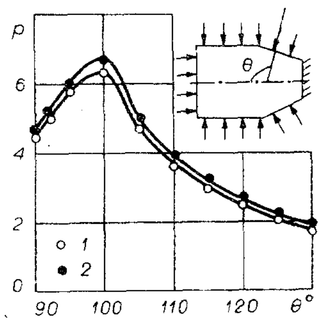

Fig. 1

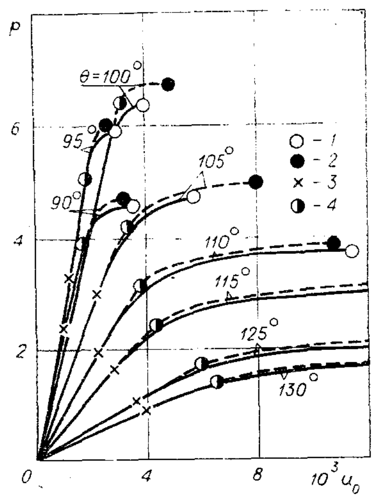

Fig. 2

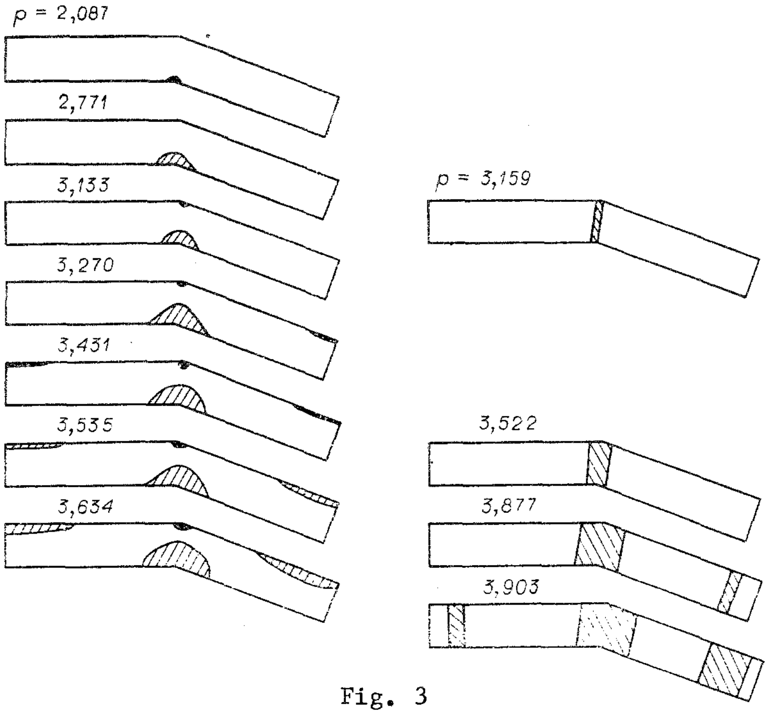

Fig. 3

In approximating the differential equations by finite differences, the cylindrical and conical parts of the shell are divided into 16 intervals each. Preliminary calculations show that if the step in the deformation parameter is chosen so that ∿10 steps are taken before plastic deformations occur, and ∿30 steps up to the limit load; a further decrease of the step size leaves the limit load practically unchanged.

The matrix elements $[B_{ij}]$ in Eq. (1.4) are calculated by both Eqs. (1.5) and Eqs. (1.6). In using Eqs. (1.5) the integrals are evaluated by Simpson's rule with the thickness of the shell divided into eight intervals.

Figure 1 shows the dependence of the limit load on the angle θ (points 1 are calculated by Eqs. (1.5) and points 2 by Eqs. (1.6)]. The graphs show that Eqs. (1.6) give higher values of the limit loads, but the maximum difference in the results does not exceed 6%.

Figure 2 shows the $p \sim u_0$ curves (u_0 is the dimensionless axial displacement of the left-hand end of the shell) calculated by Eqs. (1.5) (solid curves) and by Eqs. (1.6) (open curves); points 1 and 2 are the limit loads; 3 and 4 are the values of the pressures at which zones of plastic deformation first appear according to Eqs. (1.5) and (1.6) respectively.

According to the hypotheses under which Eqs. (1.6) were derived, a cross section of a shell is either completely in a plastic state or completely in an elastic state. The extent of zones of plastic deformations according to Eqs. (1.5) and (1.6) were compared for shells with various angles θ. Figure 3 shows the nature of the development of zones of plastic deformation for a shell with θ = 110°. The diagrams on the left-hand side of the figure were calculated with Eqs. (1.5), and those on the right-hand side by Eqs. (1.6). The calculations show that Eqs. (1.6) correctly reflect the nature of the localization of zones of plastic deformations.

Thus, comparison shows that limit loads and the dependence of the pressure on a characteristic displacement can be calculated accurately enough by Eqs. (1.6). By using relations of the form (1.6) the numerical solution of nonaxially symmetric problems of the elastic-plastic deformation of shells can be greatly facilitated.

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STABILITY OF STRUCTURE ELEMENTS SUBJECTED TO STATIONARY LOADS

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The problem of the stability of viscoelastic rods and shells subjected to compressive loads varying randomly with time was examined in [1]. The method of moment functions is used for the solution. The problem mentioned is among the class of stochastically nonlinear problems; hence, the system of equations in the desired moment functions turns out not to be closed [2-4]. Closure of the system of equations is realized by using the hypothesis that the process being studied is quasi-Gaussian, whereupon an approximate solution is obtained. A feature in the construction of such a solution makes estimation of the degree of its error quite problematical in the general case. From this viewpoint, an analysis of the exact solutions of the problems mentioned is of indubitable interest since its illustration can result in a comparison between the outcomes obtained by approximate and exact methods.

This paper is devoted to an examination of the exact method of solving problems on the stability of structure elements subjected to random loads.

We assume that a viscoelastic rod loaded by stationary transverse loads and a compressive force applied to the ends is at rest on a continuous viscoelastic foundation. The equilibrium equation for such a rod in the quasistatic formulation of the problem is

$$w = -(c + \mathbf{K})\left[(1 - \mathbf{\Gamma})EIw^{\mathrm{IV}} + P(w + w_0)'' - q\right], \tag{1}$$

where

$$\mathbf{\Gamma} f = \int_{t_0}^{t} \Gamma(t-\tau) f(\tau)\, d\tau, \quad \mathbf{K}\varphi = \int_{t_0}^{t} K(t-\tau)\varphi(\tau)\, d\tau;$$

and w, w_0 are the additional and initial rod deflections. The remaining notation is standard.

The relaxation $\Gamma(t-\tau)$ and creep $K(t-\tau)$ kernels characterize the viscous properties of the material of the rod and of the viscoelastic foundation.

Considering the rod hinge-supported at the ends and assuming

$$w_0(x) = f_0 \sin\frac{k\pi}{l}x,$$
$$w(x, t) = f(t)\sin\frac{k\pi}{l}x, \quad q(x, t) = q^0(t)\sin\frac{k\pi}{l}x,$$

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